2020 John O'Bryan Mathematics Competition :: 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper.** Questions will not be scored without the following two things:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

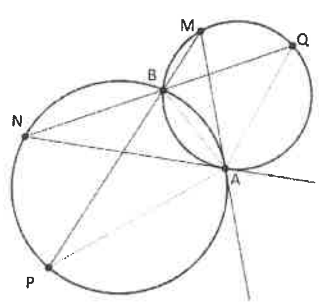
Questions are equally weighted. Teams must **show complete solutions (not just answers)** to receive credit. More specific instructions are read verbally at the beginning of the test.

1. Aliyah and Ynez both have sticks (not necessarily the same size). First, Aliyah breaks her stick into three parts, and then Ynez breaks hers into three parts. If it is possible to construct two triangles from the six pieces (i.e. each piece serves as one side in one triangle), then Ynez wins; otherwise Aliyah wins.
   (a) If the six pieces have lengths 5, 7, 12, 13, 24, and 25, show that Ynez is the winner.
   (b) Assume Aliyah's stick is 39 inches long, and Ynez's stick is 27 inches long. Show Aliyah can win.
   (c) Assume Aliyah's stick has length k and is broken into pieces of lengths $k_1 \geq k_2 \geq k_3$. Assume Ynez's stick has length ℓ with $\ell \geq k$. Show that Ynez can always win.

2. Assume x, y, z are integers with $x \geq y > 0 > z$. Consider the equation
$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3.$$
   (a) Find a solution where $|x| = |z|$ and $|x| + |y| + |z| < 10$.
   (b) Find a solution where $y = \frac{x}{2}$.
   (c) Show that there are infinitely many solutions to the equation.

3. In the circles below, note that: (1) the circles intersect at A and B; (2) $\overrightarrow{MA}$ is tangent to the circle containing P; and (3) $\overrightarrow{NA}$ is tangent to the circle containing Q.

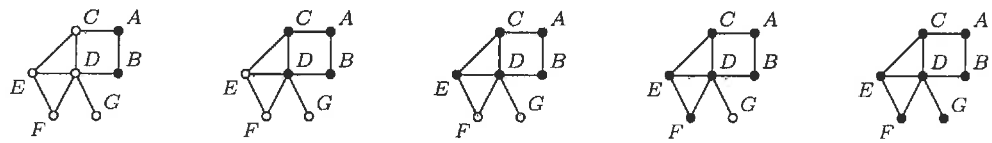
   Recall that the *Inscribed Angle Theorem* states that an inscribed angle is half the measure of the arc it intercepts (i.e. subtends). As a consequence, each of the two adjacent angles formed by a tangent and a chord drawn from the point of tangency is equal to half the measure of the arc it intercepts (i.e. subtends).



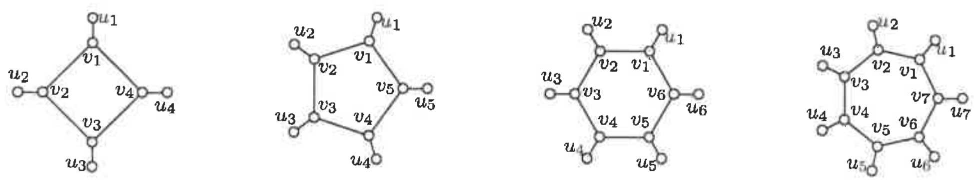
   (a) Prove that triangles AQN and AMP are similar.
   (b) Prove that $\angle ABQ = \angle MAN$.
   (c) Prove $\overline{MP} = \overline{NQ}$.

4. Let $a \neq -1$.
   (a) Calculate $\dfrac{a^5+1}{a+1}$.
   (b) Assume $a+1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.
   (c) Let $k \geq 1$ and assume $4^{5^{k-1}} + 1$ is divisible by 5^k. Show that $4^{5^k} + 1$ is divisible by 5^{k+1}.

5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, \ldots$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy neighbor, that healthy neighbor becomes (and stays) infected at the next time step. Starting (below far left) with two initially infected vertices (i.e. infected at $t = 0$), the infection takes 4 steps to infect the entire graph as follows: $A \to C$ and $B \to D$; then $C \to E$; then $E \to F$; and finally $D \to G$.



The COVID graph, C_n, has n vertices in a circle, each with a single additional neighbor; see examples of C_4, C_5, C_6, and C_7 below.

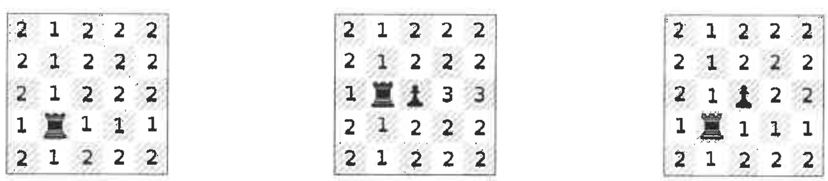


(a) If u_1, u_4 are initially infected, how long does it take to completely infect C_4?

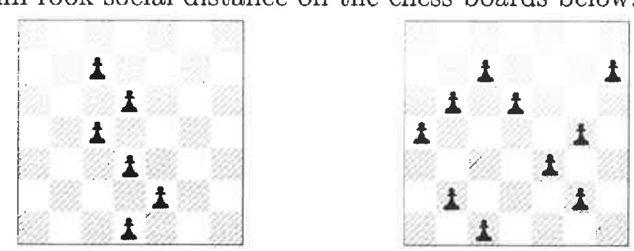
(b) For C_6 and C_7, determine the minimum number of initially infected vertices that will completely infect the graph.

(c) For C_{2019}, determine both the minimum number of initially infected vertices that will completely infect the graph, and also the number of steps it takes to do so.

6. In chess, a rook (i.e. castle) can move any number of spaces vertically or horizontally. We define the *rook social distance* between squares on a chess board as the number of moves it takes a rook to move from one to the other. If there are pawns on the chess board, social distancing dictates that the rook is not allowed to capture it. See pictures below labeling rook social distances from its position to all other squares on the chess board, and how that is altered with a pawn on the board; note that a careful choice of a "rook starting square" is sometimes required to maximize a board's rook social distance.



(a) Determine the maximum rook social distance on the chess boards below.



(b) How many ways can 3 rooks can be placed on a 3×3 chess board so that the squares they are on all have rook social distance at least 2? How about 3 rooks on a 4×4 chess board? An $n \times n$ chess board?

(c) On a 5×5 chess board, what is the minimum number of pawns required to obtain a rook social distance of 6 between two squares?

2020 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Let k be the area of a right triangle with a leg of length 36 and hypotenuse of length 85. Let w be the area of a square whose diagonal has a length of $\sqrt{242}$. Find the value of $(k+w)$.

2. If x is 80% of y and y is 25% of z, then z is what percent of x?

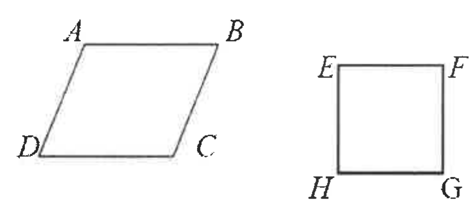
3. The product of two consecutive negative integers is 144 more than the smaller of the two integers. Find the sum of the two consecutive integers.

4. Sixty-four players stand in a circle. Player 1 stays in. Player 2 is knocked out. Player 3 stays in. Player 4 is knocked out. Player 5 stays in. Player 6 is knocked out. This process continues, knocking every other Player out, until only one player remains. What is that player's number?

5. Find the slope of a line that is perpendicular to the line that contains the points (48,0) and (0, –24).

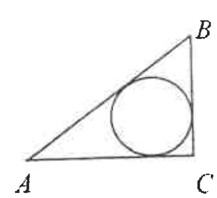
6. The repeating decimal $0.68\overline{3}$ (where only the 3 repeats) can be written as $\frac{k}{w}$ where k and w are positive integers. Find the smallest possible value of $(k+w)$.

7. In the diagram, $ABCD$ is a parallelogram which is not a rhombus. $EFGH$ is a square. $\overline{AB} \cong \overline{EF}$. If two of the eight sides are selected at random (without replacement), find the probability that the two sides selected are congruent. Express your answer as a common fraction reduced to lowest terms.



8. Jake enters a convenience store and says: "Give me as much money as I have now, and I will spend \$12 in your store." The manager agrees, and Jake spends \$12. Jake then enters a candy store and says: "Give me as much money as I have now, and I will spend \$12 in your store." The manager agrees, and Jake spends \$12. Finally, Jake enters an ice cream store and says "Give me as much money as I have now, and I will spend \$12 in your store." The manager agrees, and Jake spends \$12 after which he has no money left. Find the number of **cents** that Jake started with.

9. In the diagram, $\overline{AB}$ is the hypotenuse of ΔABC and has a length of 20. The radius of the inscribed circle has a length of 4. Find the perimeter of ΔABC.



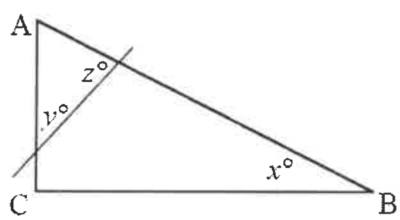
10. The smaller of two consecutive positive integers is an integral multiple of 23, and the larger of the two consecutive positive integers is an integral multiple of 29. If the smaller integer is less than 2000, find the sum of all distinct possibilities for the smaller integer.

11. Let k be the smallest integer and let w be the largest integer such that both k and w are values for x that satisfy the inequality $|2x-1|<15$. Find the value of $(3k+2w)$.

12. In rhombus $ABCD$, $\angle DAB = 60°$. A circle passes through vertices A, B, and D, and intersects diagonal $\overline{AC}$ at E. If $EC = 12$. If $k\pi$ is the area of the circle, find the value of k.

13. A triangle with base $\overline{AB}$ has a height to base $\overline{AB}$ whose length is $\frac{3}{x}$ units, $x>0$. The area of this triangle is $\frac{24}{x}$ square units. Find the number of units in the length of the base $\overline{AB}$.

14. In the figure, if $x = 3y$, $\angle ACB = 90°$, and $\angle BAC = 27°$, find the value of z.



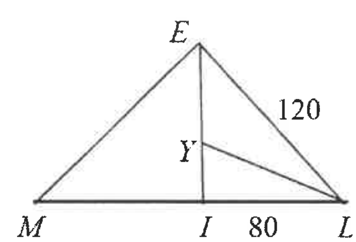
15. A circle has an equation of $(x-8)^2 + y^2 + 10y = -13$. Find the length of the radius of the circle. Give your answer in the form $a\sqrt{b}$ where both a and b are integers and b is prime.

16. A parallelogram that is **not** a rectangle has an area of 184. The two diagonals of the parallelogram divide the parallelogram into four triangles. Find the smallest possible area of any one of these four triangles.

17. Let k represent a positive integer. Let k be divided into 4 parts such that each part is a positive integer and such that the sum of the four parts is k. If the first part were increased by 4, the second part decreased by 4, the third part divided by 4, and the fourth part multiplied by 4, then the four results would be equal. Find the smallest possible value of k.

18. The length of the hypotenuse of a right triangle is 97, and the sum of the lengths of the two legs of this right triangle is 137. Find the length of the smaller leg.

19. In the diagram, M, I, and L are collinear, and E, Y, and I are collinear. Ray $\overrightarrow{LY}$ bisects $\angle ELM$. $\overline{EI}$ is an altitude of ΔEML. EL=120, and IL=80. Find the value of $(EY - YI)$. Give your answer in the form $a\sqrt{b}$ where both a and b are integers and b is prime.



20. Let a, b, and c represent single digit positive integers such that $f(x) = ax^2 + bx + c$. If $f(4) = 63$ and $f(10) = 273$, find the value of $a + 2b + 3c$.

Name: ______________________ Team Code: _______

2020 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ____________________	**11.** ____________________
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10. ____________________	**20.** ____________________

2020 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. Find the **sum** of all distinct members of the **union** $A \cup B$ of the two sets $A = \{1,2,4,5,7\}$ and $B = \{3,4,6,7,8\}$.

2. If $f(x) - x(f(-x)) = x^2$, find $f(5)$. Express your answer as an improper fraction reduced to lowest terms.

3. Find the smallest positive integer that has exactly 32 distinct positive integral divisors, but had only 3 distinct prime positive factors.

4. Let $5+\sqrt[3]{7}$ be a root for *x* of a polynomial equation in terms of *x* of the smallest possible degree in which all coefficients are integers. If the coefficient of the term of the highest degree is 1, find the sum of the numerical coefficients of the x^2 and *x* terms.

5. If *n* is a positive integer, find the value of *n* such that
$$\frac{1(2)+3(4)+5(6)+\cdots+(2n-1)(2n)}{1(2)(3)+2(3)(4)+3(4)(5)+\cdots+n(n+1)(n+2)} = \frac{29}{150}$$

6. Find the length of the minor axis of an ellipse whose equation is $27x^2 + 144y^2 = 3888$. Give your answer in the form $a\sqrt{b}$ where both *a* and *b* are integers and *a* is as large as possible.

7. Let $i = \sqrt{-1}$. If $x + 2i = y$ find the value of $|x-y| + |y-x|$

8. Find the remainder when $(11)^{(27)^6}$ is divided by 7.

9. The terms of an arithmetic sequence are: $50, 75, 100, \cdots$. The terms of the triangular sequence whose n^{th} term is $\frac{n^2+n}{2}$ are: $1, 3, 6, 10, \cdots$. Find the value of the first term of the triangular sequence that is greater than the corresponding term of the arithmetic sequence.

10. If Jayden selects 1 coin at random from *k* coins, the probability the coin will be a nickel is $\frac{1}{3}$. If Cody selects 2 coins at random and without replacement from the same *k* coins, the probability that both coins will be nickels is $\frac{1}{12}$. What is the smallest possible value of *k*?

11. Given the set: $\{\log_2(16), \log_3(16), \log_4(16), \log_5(16), \log_6(16), \log_7(16), \log_8(16)\}$. If one of the members of the set is drawn at random, find the probability that the member drawn could represent a positive integer. Express your answer as a common fraction reduced to lowest terms.

12. Find the ordered pair that represents the sum of the following two vectors: $(-5,6)$ and $(17,7)$

13. In ΔABC, find the exact length of $\overline{AC}$ if AB=20, BC=80, and $\angle ABC = 120°$. Give your answer in the form $a\sqrt{b}$ where both a and b are integers and a is as large as possible.

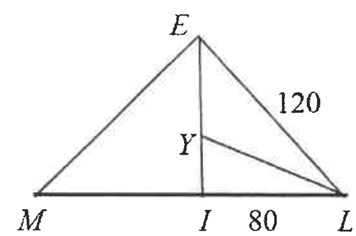
14. Alessandra is sitting in the stands behind one end-zone of a football field. She observes, in the same vertical plane with her position, two players standing on their own goal lines, exactly 100 yards apart. Looking at the spots on the ground where they are standing, a player at one goal line is at an angle of depression of 15° and a player at the other goal line is at an angle of depression of 8°. Assuming the football field is horizontal, find the **number of feet** in the vertical height of the woman's eye above the horizontal plane of the football field. Express your answer as a decimal rounded to the nearest hundredth of a foot.

15. If three real geometric means are inserted between 15 and $\frac{5}{27}$ to form the first five terms of a geometric sequence, find the value of the third term. Express your answer as an improper fraction reduced to lowest terms.

16. Let there be k objects, each of weight w. When these objects are weighed in pairs, the sum of the weights of all possible pairs is 770. When these objects are weighed in groups of four, the sum of the weights of all possible groups of four is 9240. Find the value of $(k+w)$

17. Tom is a gambler. He has a pair of fair, standard cubical dice. He selects two different numbers in advance from the face numbers 1, 2, 3, 4, 5, and 6. He then rolls the pair of dice. If both numbers showing match his two selections he wins \$108. If exactly one of his two selections is matched by either one or both dice, he neither wins nor loses. If neither number showing matches either of his two selections, he loses \$54. Find Tom's mathematical expectation each time he rolls the pair of dice. Express your answer by including the word "lose" or the word "win". For example, your answer might be "lose \$2" or "win 7".

18. In the diagram, M, I, and L are collinear, and E, Y, and I are collinear. Ray $\overrightarrow{LY}$ bisects $\angle ELM$. $\overline{EI}$ is an altitude of ΔEML. EL=120, and IL=80. Find the value of $(EY - YI)$. Give your answer in the form $a\sqrt{b}$ where both a and b are integers and b is prime.



19. In rhombus $ABCD$, $\angle DAB = 60°$. A circle passes through vertices A, B, and D, and intersects diagonal $\overline{AC}$ at E. If $EC = 12$. If $k\pi$ is the area of the circle, find the value of k.

20. Points A, B, C, and D are the vertices of a rectangle. Point E is in the interior of the rectangle, with $AE = 8$ and $EC = 21$. The lengths of $\overline{DE}$, $\overline{BE}$, and $\overline{BC}$ are integers. If $DE < BE$, find the largest possible value for the length of $\overline{BC}$.

Name: ____________________ Team Code: ______

2020 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

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2020 John O'Bryan Mathematics Competition :: 5-Person Team Test

1. Aliyah and Ynez both have sticks (not necessarily the same size). First, Aliyah breaks her stick into three parts, and then Ynez breaks hers into three parts. If it is possible to construct two triangles from the six pieces (i.e. each piece serves as one side in one triangle), then Ynez wins; otherwise Aliyah wins.

   (a) If the six pieces have lengths 5, 7, 12, 13, 24, and 25, show that Ynez is the winner.

   > **Solution:** By the Triangle Inequality, any grouping of lengths $a \le b \le c$ with $a + b > c$ forms a triangle. Also, notice that $(5, 12, 13)$ and $(7, 24, 25)$ are Pythagorean Triples that form two right triangles.

   (b) Assume Aliyah's stick is 39 inches long, and Ynez's stick is 27 inches long. Show Aliyah can win.

   > **Solution:** As a consequence of the Triangle Inequality, Aliyah wins by creating a piece longer than the sum of all the other pieces. For example, Aliyah breaks her stick into pieces of lengths 35, 2, and 2. Since every piece of Ynez's stick has length less than 27, the sum of any 2 pieces is less than 29.
   >
   > In general, let Aliyah's stick have length k and Ynez's stick have length ℓ with $k > \ell$. Assume Aliyah's stick is broken into pieces with length $\ell + \frac{2}{3}(k - \ell)$, $\frac{1}{6}(k - \ell)$, and $\frac{1}{6}(k - \ell)$ Since the piece of length $\ell + \frac{2}{3}(k - \ell)$ must be used in a triangle, the Triangle Inequality implies that the sum of the lengths of the other two sides must be longer than this. However, the sum of all 5 remaining pieces is $\ell + 2\frac{1}{6}(k - \ell) = \frac{1}{3}\ell + \frac{2}{3}k$, which means that the sum of any two will be less than what is required.

   (c) Assume Aliyah's stick has length k and is broken into pieces of lengths $k_1 \ge k_2 \ge k_3$. Assume Ynez's stick has length ℓ with $\ell \ge k$. Show that Ynez can always win.

   > **Solution:** Ynez can break her stick into pieces of lengths k_1, $\frac{1}{2}(\ell - k_1)$, and $\frac{1}{2}(\ell - k_1)$. The Triangle Inequality implies this is possible to construct an isosceles triangle from segments of length a, a, b if and only if $2a > b$.
   >
   > Because $2k_1 > k_1 > k_2$ and $2 \cdot \frac{1}{2}(\ell - k_1) = \ell - k_1 = k_2 + k_3 > k_3$, we can form two isosceles triangles with side lengths
   >
   > $$k_1, k_1, k_2 \text{ and } \frac{1}{2}(\ell - k_1), \frac{1}{2}(\ell - k_1), k_2.$$

2. Assume x, y, z are integers with $x \geq y > 0 > z$. Consider the equation

$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3.$$

(a) Find a solution where $|x| = |z|$ and $|x| + |y| + |z| < 10$.

Solution: Since $|x| = |z|$ and $x > 0 > z$, we have $z = -x$, which gives

$$2x^2 + y^2 = y^3.$$

Solving for x gives $x = \sqrt{\frac{y^3-y^2}{2}} = y\sqrt{\frac{y-1}{2}}$. Since x is an integer, $\frac{y-1}{2}$ must be a perfect square. Since $x + y + |z| < 10$ and $x \geq y$, $\frac{y-1}{2} = 1$. Therefore, $y = 3$, which makes $x = 3$ and $z = -3$.

(b) Find a solution where $y = \frac{x}{2}$.

Solution: To construct an example, when $y = \frac{x}{2}$, we have

$$\begin{aligned} 0 &= x^3 + y^3 + z^3 - x^2 - y^2 - z^2 \\ &= \frac{9x^3}{8} + z^3 - \frac{5x^2}{4} - z^2. \end{aligned}$$

To eliminate the z terms, pick $z = -x$ so that

$$\begin{aligned} 0 &= \frac{x^3}{8} - \frac{9x^2}{4} \\ &= \frac{x^2}{4}\left(\frac{x}{2} - 9\right), \end{aligned}$$

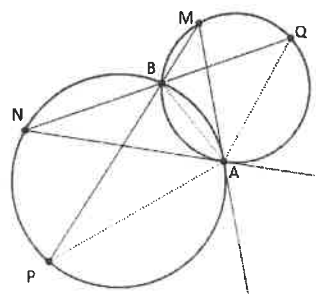
which implies $x = 18$, making $y = 9$ and $z = -18$.

(c) Show that there are infinitely many solutions to the equation.

Solution: When $z = -x$, the analysis in part (a) gives that the equation is satisfied whenever $\frac{y-1}{2} = k^2$ for any $k \in \mathbb{Z}$. Solving for y gives $y = 2k^2 + 1$, which implies $x = k(2k^2 + 1)$ and $z = -k(2k^2 + 1)$. Since k can be any integer, we have infinitely many solutions.

3. In the circles below, note that: (1) the circles intersect at A and B; (2) $\overrightarrow{MA}$ is tangent to the circle containing P; and (3) $\overrightarrow{NA}$ is tangent to the circle containing Q.

Recall that the *Inscribed Angle Theorem* states that an inscribed angle is half the measure of the arc it intercepts (i.e. subtends). As a consequence, each of the two adjacent angles formed by a tangent and a chord drawn from the point of tangency is equal to half the measure of the arc it intercepts (i.e. subtends).



(a) Prove that triangles AQN and AMP are similar.

Solution: By the Inscribed Angle Theorem, $\angle P = \angle N$ and $\angle Q = \angle M$. Combined with the angle sum formula, these imply that $\angle NAQ = \angle PAM$. Since both triangles have same angle measures, they are similar.

(b) Prove that $\angle ABQ = \angle MAN$.

Solution: Note that angle ABQ is an exterior angle of the triangle ABN; so $\angle ABQ = \angle ANB + \angle BAN$. Also note that $\angle MAN = \angle BAM + \angle BAN$. Solving this for $\angle BAN$ and substituting into the previous equality gives $\angle ABQ = \angle ANB + \angle MAN - \angle BAM$, from which the desired result follows if $\angle ANB = \angle BAM$.

To this end, consider $\widehat{AB}$ on the circle containing N and P. By the Inscribed Angle Theorem, $\widehat{AB} = 2\angle ANB$. By the consequence of the Inscribed Angle Theorem, $\widehat{AB} = 2\angle BAM$. Thus, $\angle ANB = \angle BAM$, as desired.

(c) Prove $\overline{MP} = \overline{NQ}$.

Solution: Note that if triangles AQN and AMP are congruent, then the desired result follows from the fact that MP and NQ are corresponding sides in congruent triangles. Since triangles AQN and AMP are similar by (a), it suffices to show $\overline{AQ} = \overline{AM}$.

By the consequence of the Inscribed Angle Theorem, $\widehat{AQ} = 2\angle ABQ$. Similarly, $\widehat{AM} = 2\angle MAN$. Combining these with (b) gives $\widehat{AQ} = 2\angle ABQ = 2\angle MAN = \widehat{AM}$, which implies the desired result.

4. Let $a \neq -1$.

(a) Calculate $\dfrac{a^5+1}{a+1}$.

> **Solution:** Note that $a^5 + 1 = (a+1)(a^4 - a^3 + a^2 - a + 1)$.

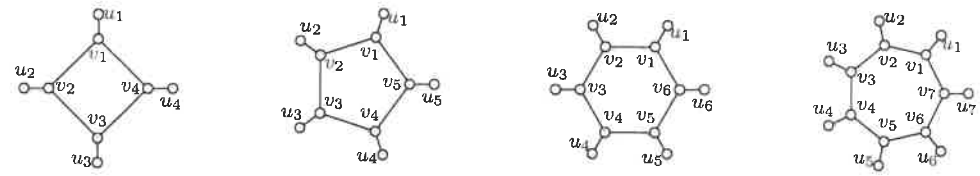
(b) Assume $a+1$ is divisible by 5. Show that $a^4 - a^3 + a^2 - a + 1$ is divisible by 5.

> **Solution:** Since 5 divides $a+1$, $a+1 = 5m$ for some $m \in \mathbb{Z}$. Thus $a = 5m - 1$. As such
>
> $$\begin{aligned} a^4 - a^3 + a^2 - a + 1 &= (5m-1)^4 - (5m-1)^3 + (5m-1)^2 - (5m-1) + 1 \\ &= (5k_4 + 1) - (5k_3 - 1) + (5k_2 + 1) - (5m - 1) + 1 \\ &= 5(k_4 - k_3 + k_2 - m) + 5 \\ &= 5(k_4 - k_3 + k_2 - m + 1), \end{aligned}$$
>
> for some $k_4, k_3, k_2 \in \mathbb{Z}$. Since $\mathbb{Z}$ is closed under addition and subtraction, $k_4 - k_3 + k_2 - m + 1 \in \mathbb{Z}$ and 5 divides $a^4 - a^3 + a^2 - a + 1$, as desired.

(c) Let $k \geq 1$ and assume $4^{5^{k-1}} + 1$ is divisible by 5^k. Show that $4^{5^k} + 1$ is divisible by 5^{k+1}.

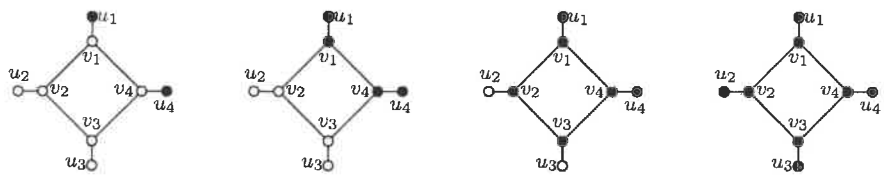
> **Solution:** Let $a = 4^{5^{k-1}}$. Since 5^k divides $a+1$, we have $a + 1 = 5^k m$ for some $m \in \mathbb{Z}$. By part (a),
>
> $$4^{5^k} + 1 = a^5 + 1 = (a+1)(a^4 - a^3 + a^2 - a + 1).$$
>
> By part (b), $a^4 - a^3 + a^2 - a + 1 = 5n$ for some $n \in \mathbb{Z}$. Thus
>
> $$4^{5^k} + 1 = 5^k m \cdot 5n = 5^{k+1} mn.$$
>
> Since $\mathbb{Z}$ is closed under multiplication, $mn \in \mathbb{Z}$. Therefore 5^{k+1} divides $4^{5^k} + 1$, as desired.

5. In discrete time steps (i.e. $t = 0, 1, 2, 3, 4, ...$), a contagion spreads around vertices in a graph where infected vertices are solid and healthy vertices are open. If an infected vertex has only one healthy neighbor, that healthy neighbor becomes (and stays) infected at the next time step. The COVID graph, C_n, has n vertices in a circle, each with a single additional neighbor; see examples of C_4, C_5, C_6, and C_7 below.



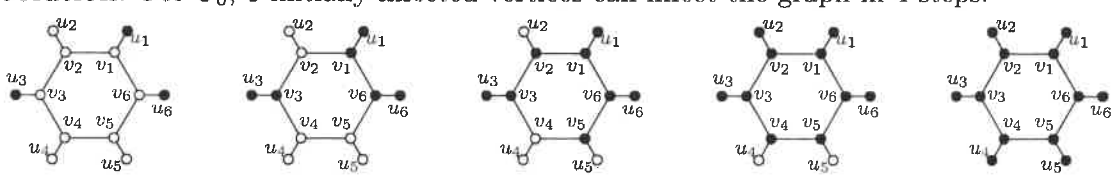
(a) If u_1, u_4 are initially infected, how long does it take to completely infect C_4?

Solution: 3 steps: $u_1 \to v_1$ and $u_4 \to v_4$; $v_1 \to v_2$ and $v_4 \to v_3$; $v_2 \to u_2$ and $v_3 \to u_3$.

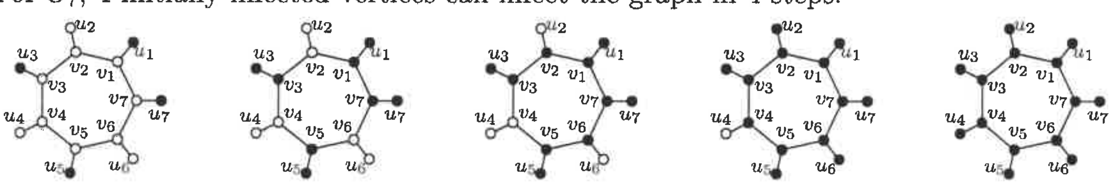


(b) For C_6 and C_7, determine the minimum number of initially infected vertices that will completely infect the graph.

Solution: For C_6, 3 initially infected vertices can infect the graph in 4 steps.



For C_7, 4 initially infected vertices can infect the graph in 4 steps.



(c) For C_{2019}, determine both the minimum number of initially infected vertices that will completely infect the graph, and also the number of steps it takes to do so.

Solution: Note (1) that an initially infected u_i will infect v_i, so we need not initially infect any v_i. Note (2) that v_i will infect u_i when $v_{i\pm1}$ is infected, and will infect $v_{i\pm1}$ when $v_{i\mp1}$ is infected. Lastly note (3) that if none of $v_{i-1}, v_i, v_{i+1}, u_{i-1}, u_i, u_{i+1}$ are initially infected, then at most v_{i-1}, v_{i+1} can be infected.

To avoid (3) and to ensure (2), initially infect u_i for all odd $i \in [2019]$ – there are 1010 such vertices. Note that (1) implies all v_i with odd i are infected on step 1; as such, with $v_{2k\pm1}$ both infected for any k, u_{2k} is infected the step after v_{2k} becomes infected. Note that (2) implies v_2, v_{2018} are infected on step 2, after which, for $k \in [505]$, $v_{2k}, v_{2020-2k}$ are infected on step $k+1$. As such, the last v_i infected is v_{1010} on step 506, after which it infects u_{1010} on step 507.

In general, C_{4m} needs $2m$ initial and $m+1$ steps; C_{4m+1} needs $2m+1$ initial and $m+2$ steps; C_{4m+2} needs $2m+1$ initial and $m+3$ steps; and C_{4m+3} needs $2m+2$ initial and $m+3$ steps.

6. In chess, a rook (i.e. castle) can move any number of spaces vertically or horizontally. We define the *rook social distance* between squares on a chess board as the number of moves it takes a rook to move from one to the other. If there are pawns on the chess board, social distancing dictates that the rook is not allowed to capture it. See pictures below labeling rook social distances from its position to all other squares on the chess board, and how that is altered with a pawn on the board; note that a careful choice of a "rook starting square" is sometimes required to maximize a board's rook social distance.

2	1	2	2	2
2	1	2	2	2
2	1	2	2	2
1	♜	1	1	1
2	1	2	2	2

2	1	2	2	2
2	1	2	2	2
1	♜	♟	3	3
2	1	2	2	2
2	1	2	2	2

2	1	2	2	2
2	1	2	2	2
2	1	♟	2	2
1	♜	1	1	1
2	1	2	2	2

(a) Determine the maximum rook social distance on the chess boards below.

Solution: 5 and 8.

3	3	3	3	3	2	2
4	4	♟	3	3	2	2
4	4	5	♟	3	2	2
4	4	♟	3	3	2	2
4	4	5	♟	3	2	2
4	4	5	5	♟	2	2
4	4	5	♟	♜	1	1

7	7	7	7	6	7	7
8	8	♟	7	6	7	♟
8	♟	4	♟	6	7	6
♟	4	4	4	5	♟	6
2	3	3	3	♟	7	6
2	♟	4	4	5	♟	6
1	♜	♟	4	5	5	5

(b) How many ways can 3 rooks can be placed on a 3×3 chess board so that the squares they are on all have rook social distance at least 2? How about 3 rooks on a 4×4 chess board? An $n \times n$ chess board?

Solution: Social distance at least 2 implies there is at most one rook in each row and column. In general there are $n(n-1)(n-2)^2$ ways to do so; note this is 6 for 3×3 and 48 for 4×4 We count the number of open squares to place a rook in the first row n. This blocks a single column so there are $n-1$ open squares to place a rook in the second row. This blocks a second column, leaving $n-2$ rows with $n-2$ open squares in which to place the third rook.

(c) On a 5×5 chess board, what is the minimum number of pawns required to obtain a rook social distance of 6 between two squares?

Solution: Note that 3 pawns is sufficient.

♜	♟	6	5	5
1	2	♟	5	5
♟	3	4	4	4
4	3	4	4	4
4	3	4	4	4

Suppose there are only 2 pawns. If they are in the same row, then the maximum rook social distance is 3 and occurs between any two squares with a pawn between them. If the pawns are not in the same row, then the worst case happens when they are immediately diagonal of each other. In this case, the two squares adjacent to both pawns have rook social distance 4.

Name: ______ANSWERS______ Team Code: ________

2020 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 1507
2. 500 Percent optional
3. -23
4. 1
5. 2
6. 101
7. 4/7 Must be this reduced fraction.
8. 1050 (cents optional)
9. 48
10. 2346
11. -4
12. 144
13. 16
14. 132
15. $2\sqrt{3}$ Must be this exact answer.
16. 46
17. 50
18. 65
19. $8\sqrt{5}$ Must be this exact answer.
20. 25

Awards Lists and Solutions to the Team Competition may be found at http://math.nku.edu/job

Name: ______ANSWERS______ Team Code: _______

2020 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 36
2. $\frac{75}{13}$ Must be this improper fraction.
3. 1080
4. 60
5. 22
6. $6\sqrt{3}$ Must be exactly this answer.
7. 4
8. 1
9. 1326
10. 9
11. $\frac{2}{7}$ Must be this reduced fraction.
12. (12,13) Must be this ordered pair.
13. $20\sqrt{21}$ Must be exactly this answer.
14. 88.67 Must be exactly this decimal.
15. $\frac{5}{3}$ Must be this improper fraction.
16. 18
17. Lose 18 Must be exactly this answer.
18. $8\sqrt{5}$ Must be exactly this answer.
19. 144
20. 19